Generalization of the Poiseuille law for one- and two-phase flow in a random capillary network

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Study of single-phase fluid flow in a three-dimensional (3D) random capillary network on a regular cubic lattice has established a simple generalization of the Poiseuille law for the total flow. Results are discussed in the light of effective-medium theory and percolation theory. Detailed examination of the behavior of such networks near percolation threshold leads to an extended model which is appropriate for phase conductivities in two-phase flow. The simple expression for conductivity when combined with pore phase occupancy distributions from a rule-based percolation approach can be used to calculate relative permeabilities in 3D networks.

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I. INTRODUCTION

The problem of describing the flow of a fluid through a porous medium is encountered in many diverse fields but is of particular interest in the petroleum industry. Although in this case the problem is usually that of multiphase flow, the basic case of a single liquid is not trivial. A long-standing problem is how to relate the transport coefficients to the microscopic geometry of the medium. There have been many different approaches to this problem; a comprehensive review is found in Dullien [1].

A simple model of a permeable rock is that of a three-dimensional (3D) simple-cubic lattice of capillary tubes with random radii. This is an idealized model but is appropriate for studying the flow properties of porous media [2]. One may question the validity of using a regular cubic lattice as a realistic model for a random porous medium. However, it has been shown [3,4] that for a lattice with coordination number of 14, there is no difference in percolation behavior between a regular and a random lattice. Moreover, for simple conductivity distributions, they show that the flow properties are almost identical. Recent results [5] demonstrate that this conclusion is valid down to a coordination number of six; below this there are significant differences. The flow through each tube is given by Poiseuille's law [6]. The application of this law to each tube in the network, along with the equation of conservation of mass at each node, leads to a set of simultaneous linear equations. Given the pressure difference across the network, these equations can be solved to yield the pressures at each node in the network and hence the total flow. This is equivalent to that of electrical current in a random resistor network, a problem that has been extensively studied [7,8]. This similarity means that we can draw on the many results from other workers in this field. Capillary models have been used extensively to study the flow properties of porous media [9,10]. There are other models for relative permeabilities, such as the immiscible-lattice-gas model [11]. Kalaydjian [12] includes the dynamics of interactions between the fluids in his analysis. Recent work based on the dense random packing of equal spheres has been successful in predicting relative permeabilities [13].

For a single tube, given the pressure and viscosity, the dimensions of the tube determine the flow. The more general problem is to establish parameters that determine the flow for a random network of such tubes. There is a simple relationship between the flow, the pressure difference across the network, fluid viscosity, network length, and cross-sectional area. The first two follow from the linear nature of the problem while the latter can be inferred from the results of percolation theory [14]. This leaves us with the question as to how the details of the tube network contribute to the total flow. We can rephrase this question by asking whether there is a characteristic parameter associated with the network and, if so, how it relates to the flow. Currently there are two approaches to this problem, based on percolation theory and effective-medium theory (EMT). In the percolation approach, a characteristic length (the percolation radius) is associated with the average flow in the network [Ambegaokar, Halperin, and Langer [15] (AHL)], and in EMT, an effective bond element conductivity is calculated directly. AHL claim that the percolation radius is the fundamental size parameter that determines the flow. Despite a convincing argument, they did not provide a rigorous proof to support this statement. This method has been revisited by several workers [10,16–19], who have shown that their model for the absolute permeability of a rock, based on the AHL model, yields good quantitative agreement with experiment. We can also consider the problem from the point of view of the tube conductivities, and EMT was developed to describe electrical conduction in a random resistor network [8] but is equally applicable to fluid flow. Each tube in the network is replaced by one with conductivity g_m, which is defined by the tube conductivity distribution [14,20]. The flow is proportional to g_m, the characteristic conductivity of the network. EMT has been used successfully to predict the permeability of sandstone [21].

In almost all of the above studies, the distribution of conductivities was limited to those of a uniform or bimodal δ function. This is because the general interest was in resistor networks composed of insulators and constant resistance conductors, which can be modeled by these distributions. Moreover, the percolation behavior of such systems was considered more important than the actual value of the flow.

In this work, we consider a wide range of tube radius distributions and, from the results of numerical simulation, propose a simple generalization of Poiseuille's law to describe the total flow Q as a function of the total pressure drop across the network, ΔP. We also investigate, using our model, the dependence of the total flow on the percolation radius and effective-medium conductivity. On comparison, it will be seen that all of the results are compatible. Moreover, we provide a simple counterexample to the AHL claim that the percolation radius completely determines the total flow. We also investigate the conductivity of networks near the percolation threshold and propose an equation that describes the flow right down to this threshold. This is then extended to a simple model for two-phase flow and a description of relative permeabilities. Two-phase flow in a network may be viewed as the flow of bicontinuous phases that are topologically intertwined. The conductivity of each phase thus depends on which portion of the network (in terms of the tube radii) that it occupies.

II. MODEL

Consider a cubic lattice of capillary tubes of dimensions $N_x \times N_y \times N_z$, which are the number of tubes in the x, y, and z directions, respectively. We assume periodic boundary conditions in the y and z directions in order to simulate a large system and eliminate surface effects. We could impose the condition of no flow at the boundaries; it has been demonstrated [20] that there is no significant difference between these two types of boundary conditions for $N_x, N_y, N_z \geq 6$. The pressure gradient is in the x direction. For a single tube of radius r_i and length l, the flow Q_i is given by Poiseuille's law [6]:

$$Q_i = \frac{\pi r_i^4 \Delta P}{8\eta l} , \tag{1}$$

where η is the viscosity and ΔP the pressure difference along the tube. At each node, we have the conservation of mass

$$\sum_{i=1}^{6} Q_i = 0 , \tag{2}$$

which is equivalent to Kirchhoff's law for a resistor network. This leads to a set of simultaneous linear equations that can be solved to give the total flow Q. We have studied this problem for the following radius distributions: triangular, uniform, cubic, log-uniform, exponential, truncated normal (TN), Rayleigh, Berman, and bimodal uniform (a complete description of each is given in the Appendix). We note that these radial distributions are related to their corresponding conductivity distributions through the equation $f(r)dr = h(g)dg$, $g = r^4$. Thus a cubic radial distribution corresponds to a uniform conductivity distribution.

In each case, we have solved for the total flow in networks typically of dimensions $20 \times 10 \times 10$, equivalent to 6100 tubes. Unless explicitly stated, it is to be assumed that all results presented are from runs on a $20 \times 10 \times 10$ network. A network is defined by the radius and length of each tube: the radius is sampled from one of the distributions (i) to (ix) (see Appendix) and all tube lengths are set equal to unity. The resulting set of linear equations was solved using a sparse matrix solver from the NAG library based on the Lanczos algorithm. The program was run on a Micro VAX, a typical run taking from two to five minutes. For a given distribution, we carried out a series of calculations and found that the error was at most 10%. In order to eliminate numerical and statistical errors, the results presented are the average of a series of runs, typically 20 to 50 runs, in each case. For such a series of calculations, the individual networks will be different, although they are all sampled from the same distribution and will have the same mean. The most important thing to note is that, by using a different distribution of tubes each time, we guarantee that we are investigating only the statistical parameters of the distribution and not a particular configuration.

We now turn to the results of these simulations; for convenience, we use the tube length a as the basic unit of length. It is a well-known result of percolation theory [22] that the conductivity of a random network is proportional to the cross-sectional area divided by the width, A/L, where $A = N_y N_z a^2$ and $L = N_x a$. Since the total flow Q must vary as $1/\eta$ and ΔP, we can write down an equation for the total flow through the network,

$$Q = \frac{\Delta P A}{\eta L} k . \tag{3}$$

As discussed above, we seek to relate k to some parameter of the network. By dimensional arguments, it is easy to show that k must have the dimensions of length squared. In fact, k is the permeability of the network as defined through Darcy's law [23]. We begin by considering characteristic lengths associated with the gross properties of the network, such as the mode, median, and mean of the tube radius distribution. In Sec. IV, we investigate characteristic lengths defined by the network near percolation threshold. Having studied the variation of flow with these quantities for the various distributions, it became apparent that the mean tube radius was the correct one. We illustrate the dependence of flow on mean radius for a variety of tube radius distributions in Fig. 1; clearly, k is proportional to $\langle \rho \rangle^4$. For convenience, we use $\kappa = k/a^2$ and $r = \langle \rho \rangle a$ in plots so that the units are dimensionless. Indeed, we can write the equation

$$k = \frac{\pi}{8} \langle \rho \rangle^4 a^2 , \tag{4}$$

where $\langle r \rangle = \langle \rho \rangle a$ is the average radius, defined by

$$\langle r \rangle = \int_{R_{\min}}^{R_{\max}} r f(r) dr . \tag{5}$$

It is important to note that the line in Fig. 1 is that *predicted* by Eq. (4) and not a fit to the data. We have found that this equation predicts the permeability to a high de-

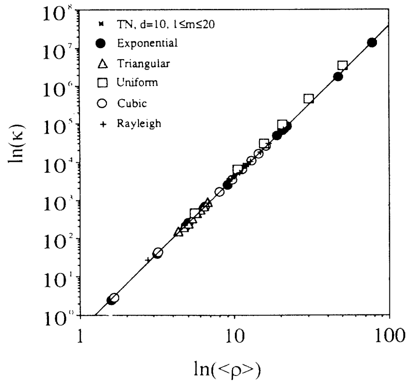

FIG. 1. Log-log plot of permeability as a function of the average tube radius for a series of distributions. The straight line is that predicted by Eq. (4), $\kappa=\pi\langle\rho\rangle^4/8$.

gree of accuracy, except for the uniform distribution. The large spread of radius values of this distribution leads to some inaccuracy; we have performed runs on larger networks, $20\times20\times20$, with significant improvement in the results (see Fig. 2). The remarkable thing is that widely differing distributions with the same mean radius have the same network permeability. The total flow through the network is given by

$$Q=\frac{\pi\langle r\rangle^4\Delta P}{8\eta L}N_yN_z\ . \tag{6}$$

We have also investigated the possibility of k being dependent on $\langle\rho^4\rangle$, another plausible generalization of Poiseuille's law. In Fig. 3, we illustrate the dependence of k on $\langle\rho^4\rangle$. We see that there is a simple linear relationship; however, there is no unique equation for all the distributions, as Eq. (4). It is clear from this result that the mean radius is the simplest parameter to characterize the flow through a random capillary network. We have also calculated k as the distribution is broadened by increasing the standard deviation σ, keeping $\langle r\rangle$ fixed. The uniform distribution is the "worst" case since it allows for the greatest variation in radius. Results are shown in Fig. 4, and it can be seen that there is little variation of k with σ.

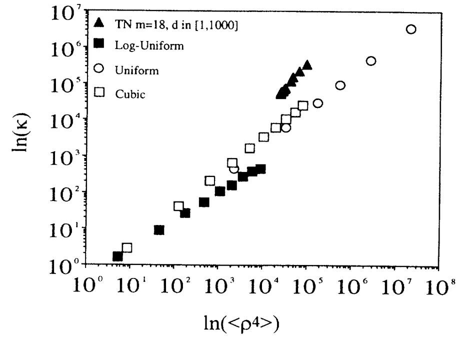

FIG. 3. Log-log plot of permeability as a function of the fourth moment of the mean tube radius for a series of radius distributions.

Poiseuille's law states that the flow is inversely proportional to the length of the tube. In order to investigate this situation in a network, we kept the radii fixed and allowed the tube *lengths* to vary according to distributions (i) to (vii) (Appendix). It was found that the flow is proportional to the reciprocal of the average tube length $\langle l\rangle$, as shown in Fig. 5. We then considered the situation where both the tube radius and length were allowed to vary, as shown in Fig. 6. This leads us to the following more general equation for the permeability of the 3D network:

$$k=\frac{\pi\langle\rho\rangle^4}{8\langle\lambda\rangle}a^2\ , \tag{7}$$

where $\langle\lambda\rangle=\langle l\rangle/a$ is the average dimensionless tube length. It is important to note that the tube radii and lengths are sampled from *different* distributions. It should also be noted that this variation in tube length is somewhat unphysical since we take it to be uncorrelated. In reality, to maintain a cubic lattice, there would be a correlation between the lengths of the tubes at adjacent nodes.

Therefore, we see that the flow through a random network of capillary tubes can be expressed as a Poiseuille law for a single tube where the radius is the average ra-

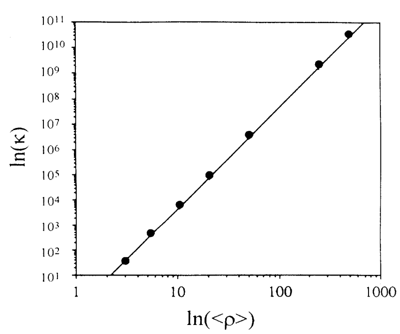

FIG. 2. Log-log plot of permeability as a function of mean radius for a uniform radius distribution on a $20\times20\times20$ network. The straight line is that predicted by Eq. (4), $\kappa=\pi\langle\rho\rangle^4/8$.

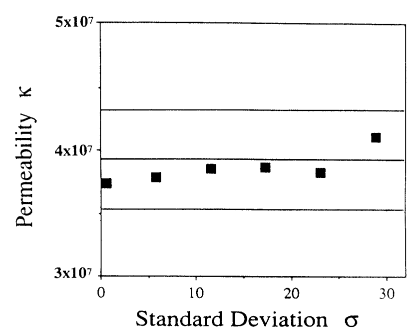

FIG. 4. Variation of permeability with the standard deviation of the radius in the case of uniform distribution, $\langle\rho\rangle=100$. The central line is the predicted value of κ; the upper and lower lines correspond to $\pm10\%$ error.

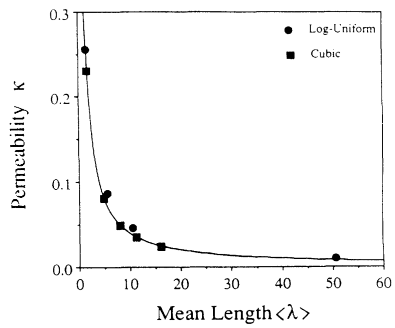

FIG. 5. Plot of the permeability against $\langle\lambda\rangle$, where $\langle\lambda\rangle$ is the mean tube length. The curve is that predicted by Eq. (7) with $\langle\rho\rangle=1$.

dius of the distribution of tubes in the network. This makes physical sense since fluid flowing through any node is more likely to encounter a tube of average size. We can also understand how tubes of average size dominate the flow if we examine the fluid velocity as a function of tube radius. We found that the highest velocities are to be found in tubes close to *average* size. Again we can argue that fluid from a given tube will flow, on average, into one of average size, implying the retardation of velocities from large tubes. This effect had already been noted in the early literature [1], but has been analyzed in detail only recently [24,25].

It is not clear how much the underlying cubic lattice influences this result: for simple conductivity distributions, the topology has little effect [3,4,5] for coordination numbers $s\geq 6$ ($s=6$ for a simple-cubic lattice). It is surprising that this law does not seem to have been observed by other workers in this area. However, in contrast to standard percolation theory, where the interest is in such quantities as the percolation threshold and cluster size distribution, we are concerned with the actual dynamics of the percolation process. This process has been investigated using effective-medium theory and percolation theory. In Secs. III and IV, we discuss our results in

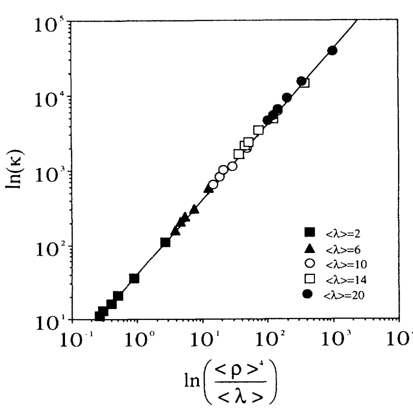

FIG. 6. Plot of the permeability against $\langle\rho\rangle^4/\langle\lambda\rangle$, where $\langle\rho\rangle$ and $\langle\lambda\rangle$ are the mean tube radius and length, respectively. The straight line is that predicted by Eq. (7), $\kappa=\pi\langle\rho\rangle^4/8\langle\lambda\rangle$. Note that the radius distribution is cubic and the length distribution log-uniform.

relation to those theories and show that they are compatible.

III. EFFECTIVE-MEDIUM THEORY

In the studies of flow in random networks, the results are usually discussed in terms of tube conductivities. The radius and conductivity distributions of a network are related by $f(r)dr=h(g)dg$ and $g=r^4$. Thus, it is easy to show that $\langle g^{1/4}\rangle$ is the characteristic quantity corresponding to $\langle r\rangle$ for a random network. However, for many years it has been known that there is another characteristic conductivity for a random network, the effective-medium conductivity. This was originally derived to model hopping conductivity in conductor-insulator materials [26,27]; Kirkpatrick [8] presents an excellent description of this method. Although it was derived for electrical current in a random resistor network, it is straightforward to apply it to fluid flow if we associate the pressure with potential difference, flow with electrical current, and flow resistance with electrical resistance [20]. Each of the tubes in the network is replaced by one with conductivity g_m, defined through the implicit equation [8,20]

$$\int_{g_{\min}}^{g_{\max}} \frac{(g-g_m)h(g)}{\left[g+\left[\frac{s}{2}-1\right]g_m\right]}dg=0\ , \tag{8}$$

where s is the coordination number of the lattice, and the total flow is given by

$$Q=\frac{\pi g_m \Delta P A}{8\eta L}\ . \tag{9}$$

Using Darcy's law, we see that $k=(\pi g_m)/8$. We have found that, for all of the distributions considered, with fixed tube length, the permeability is indeed proportional to g_m; one such example is shown in Fig. 7. From this result and that of Sec. II, it would be expected that there is a simple linear relationship between g_m and $\langle r\rangle$. Such a linear relationship was found by solving numerically between Eqs. (5) and (8), as shown in Fig. 8. This is a somewhat surprising result since there is no obvious analytic relation between the mean radius and the effective-medium conductivity.

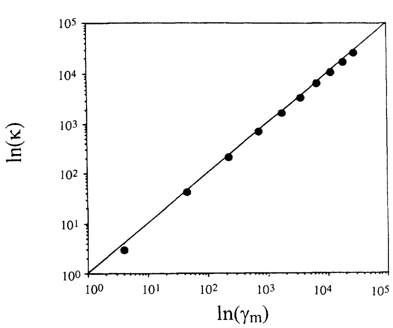

FIG. 7. Log-log plot of the permeability as a function of the effective-medium theory conductivity for a cubic radius distribution. Note that $g_m=\gamma_m a^2$.

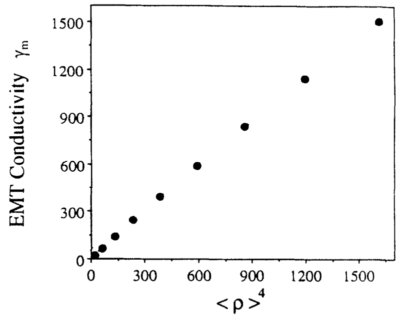

FIG. 8. Plot of the effective-medium theory conductivity as a function of the fourth power of the mean tube radius for a log-uniform radius distribution.

If a large proportion of the tube conductivities are zero (as for the log-uniform), then a large fraction of the network is nonflowing (tube radius distribution weighted heavily near zero). It is not valid to use EMT in this case since it assumes that the system is homogeneous [20,28]. We have checked to see if the same is true when our model based on the average tube radius, Eq. (4), is used. In Fig. 9, we plot k as a function of $\langle \rho \rangle$ for a log-uniform distribution of the form $h(g)=1/(2g \ln A)$, $1/A \leq g \leq A$, and zero otherwise. It is clear from Fig. 9 that the simple relation defined by Eq. (4) is not valid in this case. This also implies that EMT and our model are not valid in the vicinity of the percolation threshold.

Therefore there is close similarity between EMT and the model developed here as defined by Eq. (4); both characterize the network by an averaged quantity. We have found a numerical relationship between g_m and $\langle r \rangle$ for all the distributions considered; however, there is no obvious analytic relationship. In Eq. (8), it is clear that the effective-medium conductivity is a function of the coordination number s. There is no such relation in our model Eq. (4), and it is not clear how one could include such a dependence.

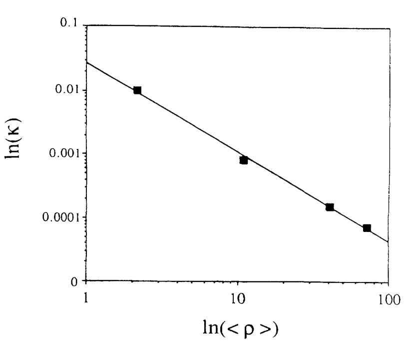

FIG. 9. Log-log plot of permeability as a function of mean radius for a log-uniform radius distribution with radii weighted heavily near zero. Note that the fitted curve is given by $0.03\langle r \rangle^{-1.39}$.

IV. PERCOLATION THEORY

Let us now consider this problem from the point of view of percolation theory; this assumes that the characteristic tube size associated with the system is closely connected to the percolation threshold [15,16]. Given a distribution of tube radii (conductivities), set all of them equal to zero and then open each of the tubes at random until flow commences. From percolation theory, we know that for a simple-cubic lattice, the fraction of bonds that needs to be replaced for flow to begin is $p_c=0.25$. This is a result for an infinite cubic lattice, and is independent of the particular radius distribution used [22]; it only depends on the geometrical structure of the network. Although we deal mainly with networks containing at most 6100 bonds, we obtain $p_c=0.25$ to a high degree of accuracy, and runs with networks up to $30\times 20\times 20$ have confirmed this result.

Opening the bonds systematically, beginning with those of largest radius, it is clear that flow will commence at a certain value of R. This is the percolation radius R_p, and is defined implicitly by the equation

$$p_c=\int_{R_p}^{R_{\max}} r f(r) dr \; . \tag{10}$$

AHL argued that the percolation radius completely characterizes the conductivity of a random resistor network [15]; if true, their argument is equally valid for a capillary tube network. They claim that just above p_c the system consists of large clusters of connected tubes. The insertion of tubes of radius R_p connects a fraction of these clusters across the network and flow begins. At this point the flow is indeed dominated by tubes of radius R_p because, no matter how large the conductivity of any cluster, the fluid must pass through single "bottleneck" tubes of radius R_p. They go on to say that even when all of the tubes in the network are open, the flow is still determined by tubes of size R_p. Given the basic geometrical difference between a cluster spanning the network and a full connected system, we were not immediately convinced by this assertion. We have investigated the variation of permeability as a function of the percolation radius. There is a simple linear relationship between k and ρ_p^4 ($R_p=\rho_p a$) for all except the TN distribution; see Fig. 10. For most of the distributions, we can write down an equation of the form

$$k=\beta \rho_p^4 a^2 \; ; \tag{11}$$

however, the constant β varies greatly between distributions, in contrast to the universality of Eq. (4). We also find that R_p is proportional to $\langle r \rangle$; for a cubic distribution with $R_{\min}=0$, it can be shown that $R_p=1.25(1-p_c)^{1/4}\langle r \rangle$, and that for a uniform distribution, $R_p=2(1-p_c)\langle r \rangle+(2p_c-1)$. For the other cases, it was necessary to solve numerically between Eqs. (10) and (14). However, in the case of the TN distribution, we found that R_p is proportional $\langle \rho \rangle^{0.54}a$.

Thus we see that, for any given radius distribution, there is a relationship between the permeability and the percolation radius; however, there is no simple universal formula as in the case of the permeability and mean ra-

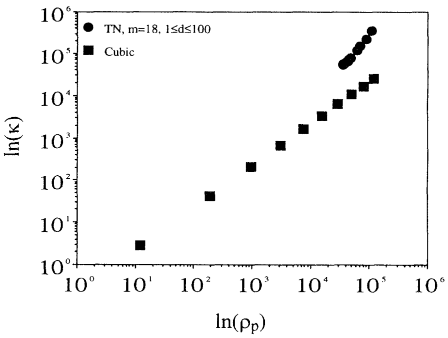

FIG. 10. Log-log plot of the permeability as a function of the percolation radius for cubic and TN radius distributions.

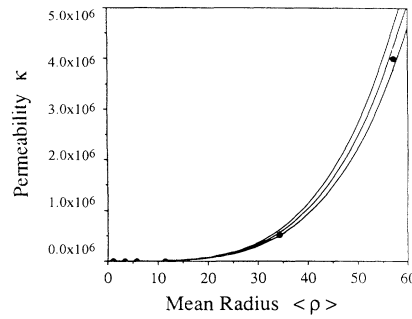

FIG. 12. Permeability as a function of the mean radius for a distribution of the form $f(r)=r^{-9}$. The central curve is the analytic prediction of Eq. (4); the upper and lower curves correspond to $\kappa \pm 10\%$ error.

dius. It should be emphasized that the lack of a simple relationship between k and R_p is for the case of the capillary network model. Another approach based on characteristic lengths resulted in very good agreement with experiment [16].

In order to investigate further the effect of R_p on the total flow, we solved the problem for a bimodal uniform distribution, as shown in Fig. 11. By placing $\geq 25\%$ of the bonds in sector (B), we ensure that the percolation radius R_p lies in that sector. We can vary the position of section (A), thus varying $\langle r \rangle$ while R_p remains constant. We find that the total flow does indeed change, demonstrating that, in some cases, the percolation radius does not completely determine the flow. This has been noted by Berman *et al.* [29], where they claim that the median conductivity determines the conductivity of two-dimensional resistor networks and discuss other counterexamples to the AHL argument. The characteristic conductivity G of any distribution must satisfy the Jensen inequality

$$\frac{1}{\langle g^{-1} \rangle} < G < \langle g \rangle \ . \tag{12}$$

The lower bound is the case of all the resistors (tubes) in series, and the upper bound of all in parallel. For a distribution of type (viii), used by Berman *et al.*, it can be shown that the percolation conductance can lie *outside* this range. However, the mean radius for such distributions is $\langle r \rangle = \frac{8}{7} R_{\min}$, and from Fig. 12 it can be seen that Eq. (4) is valid. This is another example of a distribution where the AHL argument is not valid.

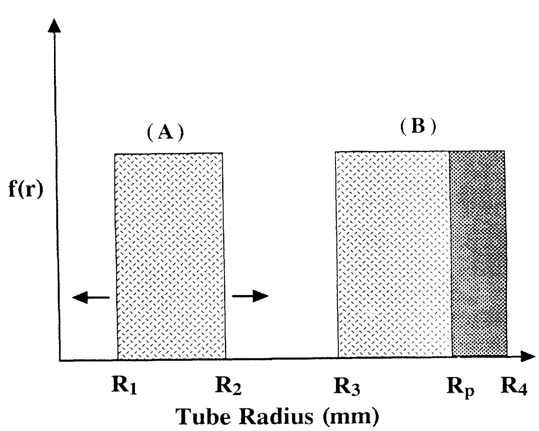

FIG. 11. Bimodal uniform radius distribution, $f(r)$. It is clear that the mean radius may be varied while the percolation radius remains fixed.

V. BEHAVIOR NEAR PERCOLATION THRESHOLD—TWO-PHASE FLOW

The conductance of networks near p_c has been studied in great detail by many authors [8,30–33]. We are interested in this region since it allows us to construct a very simple model of two-phase flow. We know that in a permeable rock saturated with two liquids, the nonwetting fluid (oil) tends to occupy the larger pores and the wetting fluid (water) is more confined to the smaller pores [23]. We model two phases in the following manner: initially the network is completely saturated with water, then oil is added to the larger pores until the percolation threshold for oil is attained. Mathematically we solve the flow equations for the oil and water bonds separately, allowing no interaction. This is indeed a very simple model, but it does produce some very interesting results.

Let us begin by considering the case of approaching the percolation threshold at random. This means that initially all of the tubes are closed, and an increasing percentage of tubes are opened at random (i.e., the order of opening having no tube radius preference) until the network is complete. The same problem has been studied in the context of resistor [8] and in capillary networks [31].

Before discussing the conductivity of such systems, it is interesting to consider the clustering behavior as the tubes are opened at random. A cluster is an isolated tube or any collection of connected tubes; a quantity of particular interest is the total number of clusters in the network. A modified Hoshen-Kopelman algorithm [34] was used to follow the clustering process. In Fig. 13, we plot the variation of the total number of clusters as a function of p, the fraction of replaced tubes. The overall form of this graph is the same for all of the distributions considered. We note that by the time the percolation threshold is reached there is a substantial decrease in the total number of clusters. This behavior can be explained in the

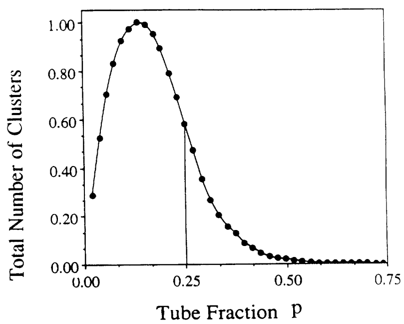

FIG. 13. Total number of tube clusters as a function of the fraction of open tubes for a uniform distribution of tube radii on a 20×20×20 network.

following manner: initially, as tubes are opened, N_{max} increases until a point of saturation is reached. The single bonds begin to aggregate to form extended clusters until there is a cluster that crosses the entire network, called a spanning cluster, at p_c. In the case of an infinite system there is a unique spanning cluster [22], but in a finite system it is possible that there may be more than one. We have never found more than one spanning cluster over many hundreds of calculations in large networks. After p_c, the system is dominated by the spanning cluster with few isolated clusters although, in a finite system, it is possible for a spanning cluster to form slightly above or below p_c.

Let us denote by $M(p)$ the fraction of bonds in the spanning cluster. It is well known [35,36] from percolation theory that $M(p)$ scales as $(p-p_c)^{\gamma}$, where $\gamma \approx 0.45$; see Fig. 14. If $n(s)$ denotes the number of clusters of size s, then it scales [35,36] as $s^{-\tau}$, where $\tau \approx 2.2$. The corresponding conductance of the network scales as $(p-p_c)^t$, and there seems to be some uncertainty relating to the value of this exponent. From percolation theory, we expect it to be a universal quantity [35], at least in the vicinity of the percolation threshold, say $p_c \le p \le p_c + 0.2$. We list the various values ascribed to this exponent in Table I. We have found a value of 1.6 ± 0.1 for randomly filled networks, in accordance with many previous results for a variety of distributions; this is true both in the immediate vicinity of p_c and over the whole range $p_c \le p \le 1$. Since

TABLE I. This contains the various values proposed for the critical conductivity exponent t in the equation $k \sim (p-p_c)^t$.

Method of derivation	Critical exponent
Computer simulation	1.6 ± 0.1,[a] 1.87[b]
Experimental	2.0,[c] 1.9 ± 0.2[d]
Theoretical	1.95 ± 0.3[e]

[a]S. Kirkpatrick, Rev. Mod. Phys. **45**, 574 (1973); J. P. Straley, Ann. Isr. Phys. Soc. **5**, 353 (1983); I. Webman, J. Jortner, and M. H. Cohen, Phys. Rev. B **16**, 2593 (1977).
[b]P. M. Adler and H. Brenner, Ann. Rev. Fluid. Mech. **20**, 35 (1988).
[c]D. Adler, L. P. Flora, and S. D. Senturia, Solid State Commun. **12**, 9 (1973).
[d]B. Abeles, H. L. Pinch, and J. I. Gittleman, Phys. Rev. Lett. **35**, 247 (1975).
[e]R. Fisch and A. B. Harris, Phys. Rev. B **18**, 416 (1978).

the tubes are opened at random, it is expected that the mean tube radius of the spanning cluster will be equal to that of the entire network and, indeed, we have confirmed this numerically.

On this basis, we propose the following extension of Eq. (7) to describe the permeability:

$$k = \frac{\pi}{8}\langle \rho \rangle^4 a^2 \left[\frac{p-p_c}{1-p_c} \right]^t , \tag{13}$$

where $t = 1.6 \pm 0.1$. Here we are carrying the averaging model right down to the percolation threshold. This relation is illustrated in Fig. 15 for a cubic radius distribution where the calculations are restricted to a region up to 20% above p_c. It may be considered incorrect to assert that $\langle r \rangle$ dominates the flow near p_c, since this is where the AHL argument is certainly valid. However, in the case of a random fill, we expect that the characteristic radius will be the mean radius. Other work in this field [3–5] indicates that results would be very similar for a random lattice structure.

For a finite system, there are bound to be large fluctuations in conductivity as we approach p_c. In such a system, the smallest bond in the spanning cluster will determine conductivity. The effect will be marked for a uni-

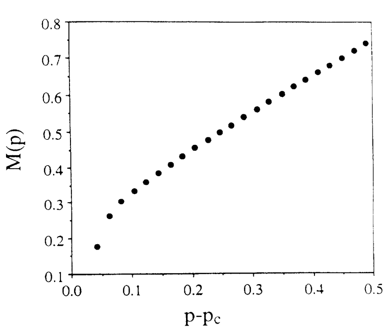

FIG. 14. Fraction of tubes in the spanning cluster as a function of $p-p_c$ for a uniform radius distribution with $\langle r \rangle = 10.5$ on a 20×20×20 network.

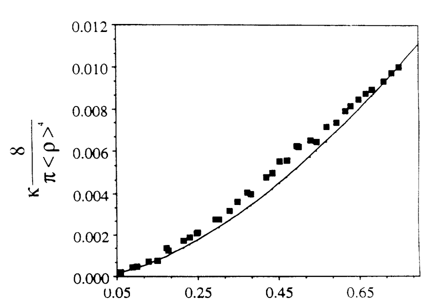

FIG. 15. Permeability divided by the factor $\pi\langle \rho \rangle^4/8$ as a function of $p-p_c$. This is for a cubic radius distribution with $\langle r \rangle = 16$. Note that the fitted value of the exponent is $t = 1.52$, whereas the curve is that predicted with $t = 1.6$.

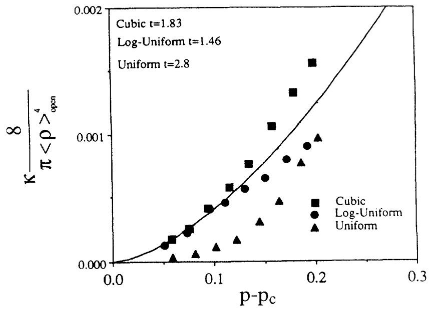

FIG. 16. Permeability divided by the factor $\pi\langle\rho\rangle^4/8$ as a function of $p-p_c$ in the case of top filling. The curve is that predicted with Eq. (15), and the fitted value of t for each distribution is included.

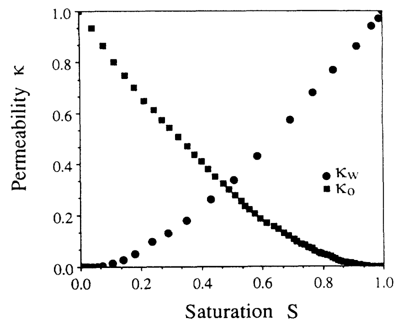

FIG. 18. Permeability as a function of saturation for a TN distribution with $m=12$ and $d=10$. The saturation S (dimensionless) is the fraction of volume that is occupied by the wetting (nonwetting) fluid. κ_o is the permeability of the nonwetting fluid and κ_w that of the wetting fluid [calculated using Eq. (15)].

form distribution, since it allows each bond to have a wide range of radii with equal probability. We would expect less statistical variation for a more peaked distribution, such as the TN, cubic, etc. In contrast to conductance, the cluster behavior is independent of the distribution considered; we found the same results in all cases for $M(p)$, $n(s)$, and $N_{\max}$.

Let us now consider in detail the cases of opening the tubes beginning with the largest, which we refer to as top filling; this is the AHL method. At each stage, we can solve the network equations for the open and closed tubes, the open tubes being filled by the nonwetting phase (oil) and the closed tubes by the wetting phase (water). At each step the permeability of each phase can be calculated, which allows for the derivation of a set of relative permeabilities [1]. This simple model allows no interaction between the two phases and also allows isolated tubes to be opened, which is not physically realistic. However, this rule-based pore-occupancy model is the simplest of a class of such models that are more physically realistic, and it is adequate to test the conductivity expressions for relative permeability that are proposed here. We can also do the converse; assume all of the tubes are filled by oil, and water enters the network beginning with the smallest tubes. We call this bottom filling.

It was evident from initial calculations that the simple scaling behavior in Eq. (13) does not apply in this case. If

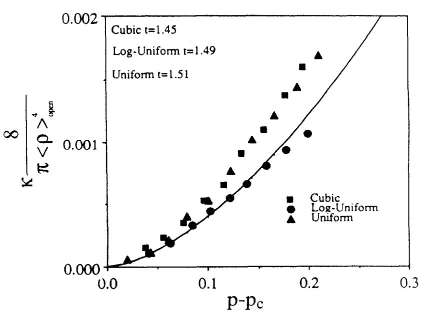

FIG. 17. Permeability divided by the factor $\pi\langle\rho\rangle^4/8$ as a function of $p-p_c$ in the case of bottom filling. The curve is that predicted with Eq. (15), and the fitted value of t for each distribution is included.

we take the case of top filling, then it is clear that the average tube radius is decreasing with decreasing p. We expected the permeability to be dominated by the mean tube radius of the *flowing cluster.* However, we found that there is no significant difference between the mean radius of the opened tubes and those in the spanning and flowing clusters. The mean radius of the opened tubes is given by

$$\langle r\rangle_{\text{open}}=\int_R^{R_{\max}} rf(r)dr \; , \tag{14}$$

where R is the minimum radius of opened tubes. This leads us to propose the following as the equation for the permeability as the tubes are opened from the top down (or bottom up):

$$k=\frac{\pi}{8}\langle\rho\rangle_{\text{open}}^4 a^2\left[\frac{p-p_c}{1-p_c}\right]^t . \tag{15}$$

In Figs. 16 and 17, we plot the results of top and bottom filling for a variety of radius distributions. As can be seen, there are discrepancies between the values of the conductivity exponent t, especially for the uniform distributions. It is difficult to evaluate the influence of finite-size effects on these results.

With the results of this simulation, we can plot relative permeability curves on the basis of this simple model; see Fig. 18. It is interesting that these curves have the same general characteristics as real relative permeability curves [37], despite the fact that there is no interaction between the two phases.

VI. CONCLUSION

We have investigated the flow properties of random capillary networks on a cubic lattice and we have found the result that the flow or permeability can be expressed quantitatively in a generalized form of Poiseuille's law. This expression is independent of the type of tube radius distribution. It should be noted that while we can write down similar equations for the flow in terms of the percolation radius, there will be largely differing constants according to the form of the radius distribution. This result was shown to be compatible with those of effective-medium theory and percolation theory. Moreover, it has been demonstrated that percolation theory does not always provide a complete description of network flow. Apart from the elegance of this result, it is very practical since calculations on large networks require much computer time, while this formula allows us to calculate directly the total flow for any given network.

It is possible to use this model to describe the flow properties of these systems right down to percolation threshold. Moreover, we have presented a simple model for two-phase flow that results in analytic expressions for the relative permeabilities.

As mentioned above, our studies were limited to simulations on a cubic lattice. It is difficult to infer how our results would look on a random lattice with coordination numbers $s=6$. However, we have pointed out that other workers in this area have shown that, for $s\geq 6$, there is little difference in percolation behavior and flow properties (for simple conductivity distributions) between random and regular lattices. It is indeed fortuitous that the relation between the flow and the network parameters in Eq. (4) takes this simple form on a cubic lattice when $s=6$.

In conclusion, we have seen that, despite their complicated nature, the overall macroscopic conductivity properties of three-dimensional random capillary networks on a regular cubic lattice can be described by very simple equations.

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APPENDIX

We present a list of the tube radius distributions used in the calculations.

(i) Triangular,

$$f(r)=\begin{cases}\alpha(r-R_{\min}), & R_{\min}\leq r\leq c\ ,\\ \beta(R_{\max}-r), & c\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ ,\end{cases}$$

where $c=(\alpha R_{\min}-\beta R_{\max})/(R_{\min}+R_{\max})$.

(ii) Uniform,

$$f(r)=\begin{cases}\dfrac{1}{(R_{\max}-R_{\min})}, & R_{\min}\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ .\end{cases}$$

(iii) Cubic,

$$f(r)=\begin{cases}\dfrac{4r^3}{(R_{\max}^4-R_{\min}^4)}, & R_{\min}\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ .\end{cases}$$

(iv) Log-uniform,

$$f(r)=\begin{cases}\dfrac{1}{Nr}, & R_{\min}\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ ,\end{cases}$$

where $N=\ln(R_{\max}/R_{\min})$.

(v) Exponential,

$$f(r)=\begin{cases}\dfrac{e^{-r}}{e^{-R_{\min}}-e^{-R_{\max}}}, & R_{\min}\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ .\end{cases}$$

(vi) Rayleigh,

$$f(r)=\frac{1}{2d^2}(r-R_{\min})e^{-d^2(r-R_{\min})^2},\quad R_{\min}\leq r\leq\infty\ ,$$

where d is a constant.

(vii) Truncated normal,

$$f(r)=\begin{cases}N(R_{\max}-r)(r-R_{\min})e^{-(r-m)^2/2d^2},\\ R_{\min}\leq r\leq R_{\max}\ ,\\ 0 \ \text{otherwise}\ ,\end{cases}$$

where m and d are constants and N is a normalization factor. This is just a normal distribution with cutoff points at $R_{\min}$ and $R_{\max}$; such distributions are similar to real pore-size distributions in porous media.

(viii) Distribution used by Berman [20],

$$f(r)=\begin{cases}\dfrac{8R_{\min}^9}{r^9}, & R_{\min}\leq r<\infty\ ,\\ 0 \ \text{otherwise}\ .\end{cases}$$

(ix) Bimodal uniform,

$$f(r)=\begin{cases}\dfrac{\alpha}{(R_2-R_1)}, & R_1\leq r\leq R_2\ ,\\ \dfrac{\beta}{(R_4-R_3)}, & R_3\leq r\leq R_4\ ,\end{cases}$$

where $\alpha+\beta=1$; see Fig. 11.

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